

EE3302-501-HW6 Solutions

Problem A-1

Difference Equation:

$$y(n) + \frac{1}{2}y(n-1) = x(n)$$

Initial Condition: $y(-1) = 0$

Input: $x(n) = \left(-\frac{1}{2}\right)^n u(n)$

(i) Impulse Response: Assume $x(n) = \delta(n)$ and $y(n) = 0$ for $n < 0$. Step 1-Homogeneous solution:

$$y(n) + \frac{1}{2}y(n-1) = 0$$

Characteristic equation:

$$\lambda + \frac{1}{2} = 0 \quad \Rightarrow \quad \lambda = -\frac{1}{2}$$

$$y_H(n) = C \left(-\frac{1}{2}\right)^n u(n).$$

Step 2-Particular solution: Assume $y_p(n) = A\delta(n)$:

$$A\delta(n) + \frac{1}{2}A\delta(n-1) = \delta(n)$$

At $n = 1$, $A = 0$, hence:

$$y_p(n) = 0$$

Total response:

$$y(n) = C \left(-\frac{1}{2}\right)^n u(n)$$

Using $n = 0$:

$$C = 1$$

$$\boxed{h(n) = \left(-\frac{1}{2}\right)^n u(n)}$$

(ii) Zero-State Response: Assume:

$$y_{zs}(n) = y_H(n) + y_p(n)$$

So:

$$y_{zs}(n) = C \left(-\frac{1}{2}\right)^n + An \left(-\frac{1}{2}\right)^n$$

Then apply:

$$y(-1) = 0 \Rightarrow C = 0$$

So:

$$y_{zs}(n) = An \left(-\frac{1}{2}\right)^n u(n).$$

Substitute into the equation:

$$An \left(-\frac{1}{2}\right)^n u(n) + \frac{1}{2}A(n-1) \left(-\frac{1}{2}\right)^{n-1} u(n-1) = \left(-\frac{1}{2}\right)^n u(n).$$

At $n = 1$:

$$-\frac{A}{2} = -\frac{1}{2} \Rightarrow A = 1$$

$$y_{zs}(n) = n \left(-\frac{1}{2}\right)^n u(n)$$

(iii) Zero-Input Response:

$$y_{zi}(n) = C_1 \left(-\frac{1}{2}\right)^n u(n)$$

(iv) Total Response:

$$y(n) = C_1 \left(-\frac{1}{2}\right)^n u(n) + n \left(-\frac{1}{2}\right)^n u(n)$$

Using $n = 0$:

$$C_1 = 1$$

$$y(n) = \left(-\frac{1}{2}\right)^n u(n) + n \left(-\frac{1}{2}\right)^n u(n)$$

Problem A-2

Difference Equation:

$$y(n) + \frac{1}{2}y(n-1) = x(n)$$

Initial Condition: $y(-1) = 1$

Input: $x(n) = \left(\frac{1}{2}\right)^n u(n)$

(i) Impulse Response: Set $x(n) = \delta(n)$ and assume $y(n) = 0$ for $n < 0$. Homogeneous solution:

$$\lambda + \frac{1}{2} = 0 \Rightarrow \lambda = -\frac{1}{2}$$

$$y_H(n) = C \left(-\frac{1}{2}\right)^n u(n)$$

Particular solution: $y_p(n) = 0$. Using $n = 0$: $y(0) = 1 \Rightarrow C = 1$

$$h(n) = \left(-\frac{1}{2}\right)^n u(n)$$

(ii) **Zero-State Response** Particular part: Since $\frac{1}{2}$ is not a root, assume:

$$y_p(n) = A \left(\frac{1}{2}\right)^n u(n)$$

Substitute:

$$A \left(\frac{1}{2}\right)^n + \frac{1}{2} A \left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^n$$

Simplify:

$$A + A = 1 \Rightarrow A = \frac{1}{2}$$

$$y_p(n) = \frac{1}{2} \left(\frac{1}{2}\right)^n u(n)$$

Homogeneous part:

$$y_{zs}(n) = C \left(-\frac{1}{2}\right)^n + \frac{1}{2} \left(\frac{1}{2}\right)^n$$

Using $y(-1) = 0$: $y(0) = x(0) - \frac{1}{2}(0) = 1$:

$$C + \frac{1}{2} = 1 \Rightarrow C = \frac{1}{2}.$$

Therefore:

$$y_{zs}(n) = \left[\frac{1}{2} \left(-\frac{1}{2}\right)^n + \frac{1}{2} \left(\frac{1}{2}\right)^n \right] u(n)$$

(iii) **Zero-Input Response:**

$$y_{zi}(n) = C \left(-\frac{1}{2}\right)^n u(n)$$

Using $y(-1) = 1$:

$$C \left(-\frac{1}{2}\right)^{-1} = 1 \Rightarrow -2C = 1 \Rightarrow C = -\frac{1}{2}$$

$$y_{zi}(n) = -\frac{1}{2} \left(-\frac{1}{2}\right)^n u(n)$$

(iv) **Total Response:**

$$y(n) = \frac{1}{2} \left(\frac{1}{2}\right)^n u(n)$$

Problem A-3

Difference Equation:

$$y(n) + \frac{8}{15}y(n-1) + \frac{1}{15}y(n-2) = x(n), \quad n \geq 0$$

Initial Conditions: $y(-1) = y(-2) = 1$

Input: $x(n) = \left(\frac{1}{3}\right)^n u(n)$

(i) Impulse Response: Assume $x(n) = \delta(n)$ and $y(n) = 0$ for $n < 0$.

Step 1: Homogeneous solution

$$15\lambda^2 + 8\lambda + 1 = 0$$

$$(5\lambda + 1)(3\lambda + 1) = 0$$

$$\lambda = -\frac{1}{5}, \quad -\frac{1}{3}$$

$$y_H(n) = \left[C_1 \left(-\frac{1}{5}\right)^n + C_2 \left(-\frac{1}{3}\right)^n \right] u(n)$$

Step 2: Particular solution

$$y_p(n) = 0$$

Total response:

$$y(n) = C_1 \left(-\frac{1}{5}\right)^n u(n) + C_2 \left(-\frac{1}{3}\right)^n u(n)$$

At $n = 0$: $y(0) = \delta(0) - \frac{8}{15}(0) - \frac{1}{15}(0) = 1$:

$$C_1 + C_2 = 1$$

At $n = 1$: $y(1) = \delta(1) - \frac{8}{15}(1) - \frac{1}{15}(0) = -8/15$:

$$-\frac{C_1}{5} - \frac{C_2}{3} = -\frac{8}{15}$$

Solving:

$$C_1 = -\frac{3}{2}, \quad C_2 = \frac{5}{2}$$

$$h(n) = -\frac{3}{2} \left(-\frac{1}{5}\right)^n u(n) + \frac{5}{2} \left(-\frac{1}{3}\right)^n u(n)$$

(ii) Zero-State Response: Assume:

$$y_p(n) = A \left(\frac{1}{3}\right)^n u(n)$$

Substituting into the difference equation at $n = 2$:

$$A \left(\frac{1}{9}\right) + \frac{8}{15}A \left(\frac{1}{3}\right) + \frac{1}{15}A(1) = \frac{1}{9},$$

therefore, $A = \frac{5}{16}$ resulting in:

$$y_{zs}(n) = \frac{5}{16} \left(\frac{1}{3}\right)^n u(n)$$

(iii) Zero-Input Response

$$y_{zi}(n) = C_1 \left(-\frac{1}{5}\right)^n + C_2 \left(-\frac{1}{3}\right)^n$$

Using $x(n) = 0$ and $y(-1) = y(-2) = 1$:

$$y(0) = 0 - \frac{8}{15}(1) - \frac{1}{15}(1) = -3/5$$

$$y(1) = 0 - \frac{8}{15}(-3/5) - \frac{1}{15}(1) = \frac{24}{75} - \frac{5}{75} = 19/75$$

Solving for C_1, C_2 with $y_H(0) = -3/5$ and $y_H(1) = 19/75$ gives:

$$C_1 = \frac{2}{5}, \quad C_2 = -1$$

$$y_{zi}(n) = \frac{2}{5} \left(-\frac{1}{5}\right)^n - \left(-\frac{1}{3}\right)^n$$

(iv) Total Response

$$y(n) = C_1 \left(-\frac{1}{5}\right)^n + C_2 \left(-\frac{1}{3}\right)^n + \frac{5}{16} \left(\frac{1}{3}\right)^n$$

Using initial conditions to evaluate $y(0)$ and $y(1)$:

At $n = 0$: $y(0) = x(0) - \frac{8}{15}y(-1) - \frac{1}{15}y(-2) = 1 - \frac{8}{15} - \frac{1}{15} = 2/5$.

$$C_1 + C_2 + \frac{5}{16} = \frac{2}{5} \implies C_1 + C_2 = \frac{7}{80}$$

At $n = 1$: $y(1) = x(1) - \frac{8}{15}y(0) - \frac{1}{15}y(-1) = \frac{1}{3} - \frac{16}{75} - \frac{5}{75} = 4/75$.

$$-\frac{1}{5}C_1 - \frac{1}{3}C_2 + \frac{5}{48} = \frac{4}{75} \implies -\frac{1}{5}C_1 - \frac{1}{3}C_2 = -\frac{61}{1200}$$

Solving this system yields $C_1 = -13/80$ and $C_2 = 1/4$.

$$y(n) = -\frac{13}{80} \left(-\frac{1}{5}\right)^n + \frac{1}{4} \left(-\frac{1}{3}\right)^n + \frac{5}{16} \left(\frac{1}{3}\right)^n$$

Problem B

Difference Equation:

$$y(n) + \frac{8}{15}y(n-1) + \frac{1}{15}y(n-2) = x(n), \quad n \geq 0$$

Initial Conditions: $y(-1) = y(-2) = 1$

Input: $x(n) = \left(\frac{1}{3}\right)^n u(n)$

(i) BIBO Stability: Characteristic equation:

$$15\lambda^2 + 8\lambda + 1 = 0 \Rightarrow (5\lambda + 1)(3\lambda + 1) = 0$$

$$\lambda_1 = -\frac{1}{5}, \quad \lambda_2 = -\frac{1}{3}$$

Since $|\lambda_1| < 1$ and $|\lambda_2| < 1$:

The system is BIBO stable.

(ii) Impulse Response: Assume $x(n) = \delta(n)$ and $y(n) = 0$ for $n < 0$.

$$y_H(n) = \left[C_1 \left(-\frac{1}{5}\right)^n + C_2 \left(-\frac{1}{3}\right)^n \right] u(n)$$

At $n = 0$: $y(0) = \delta(0) - \frac{8}{15}(0) - \frac{1}{15}(0) = 1$:

$$C_1 + C_2 = 1$$

At $n = 1$: $y(1) = \delta(1) - \frac{8}{15}(1) - \frac{1}{15}(0) = -8/15$:

$$-\frac{C_1}{5} - \frac{C_2}{3} = -\frac{8}{15}$$

Solving:

$$C_1 = -\frac{3}{2}, \quad C_2 = \frac{5}{2}$$

$$h(n) = -\frac{3}{2} \left(-\frac{1}{5}\right)^n u(n) + \frac{5}{2} \left(-\frac{1}{3}\right)^n u(n)$$

(iii) Zero-Input Response

$$y_{zi}(n) = C_1 \left(-\frac{1}{5}\right)^n + C_2 \left(-\frac{1}{3}\right)^n$$

Using $x(n) = 0$ and $y(-1) = y(-2) = 1$:

$$y(0) = 0 - \frac{8}{15}(1) - \frac{1}{15}(1) = -3/5$$

$$y(1) = 0 - \frac{8}{15}(-3/5) - \frac{1}{15}(1) = \frac{24}{75} - \frac{5}{75} = 19/75$$

Solving for C_1, C_2 with $y_H(0) = -3/5$ and $y_H(1) = 19/75$ gives:

$$C_1 = \frac{2}{5}, \quad C_2 = -1$$

$$y_{zi}(n) = \frac{2}{5} \left(-\frac{1}{5}\right)^n - \left(-\frac{1}{3}\right)^n$$

(iv) **Zero-State Response:** Assume:

$$y_{zs}(n) = A \left(\frac{1}{3}\right)^n u(n)$$

Substitute:

$$A + \frac{8}{15}(3A) + \frac{1}{15}(9A) = 1$$

$$A + \frac{24}{15}A + \frac{9}{15}A = 1 \Rightarrow \frac{48}{15}A = 1 \Rightarrow A = \frac{5}{16}$$

$$y_{zs}(n) = \frac{5}{16} \left(\frac{1}{3}\right)^n u(n)$$