

Homework 8 Solutions

Soliman Book

$$8.1. (a) X(z) = \sum_{n=-\infty}^{-1} (-3)^n z^{-n} = \frac{-1}{1+3z^{-1}} = \frac{-z}{z+3}$$

$$ROC: |z| < 3$$

$$(b) X(z) = \sum_{n=-5}^5 x(n) z^{-n}$$

$$= z^5 + z^4 + z^3 + \dots + z^{-4} + z^{-5}$$

$$ROC: 0 \leq |z| < \infty$$

$$(c) X(z) = \sum_{n=-\infty}^{-1} 3^n z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n z^{-n}$$

$$= \frac{-1}{1-3z^{-1}} + \frac{1}{1-\frac{1}{3}z^{-1}} = \frac{z}{z-\frac{1}{3}} - \frac{z}{z-3}$$

First term converges for $|z| > \frac{1}{3}$, second $|z| < 3$

$$ROC: \frac{1}{3} < |z| < 3$$

$$(d) \quad X(z) = 2 - \sum_{n=0}^{\infty} 2^n z^{-n} = 2 - \frac{1}{1-2z^{-1}}$$

(2)

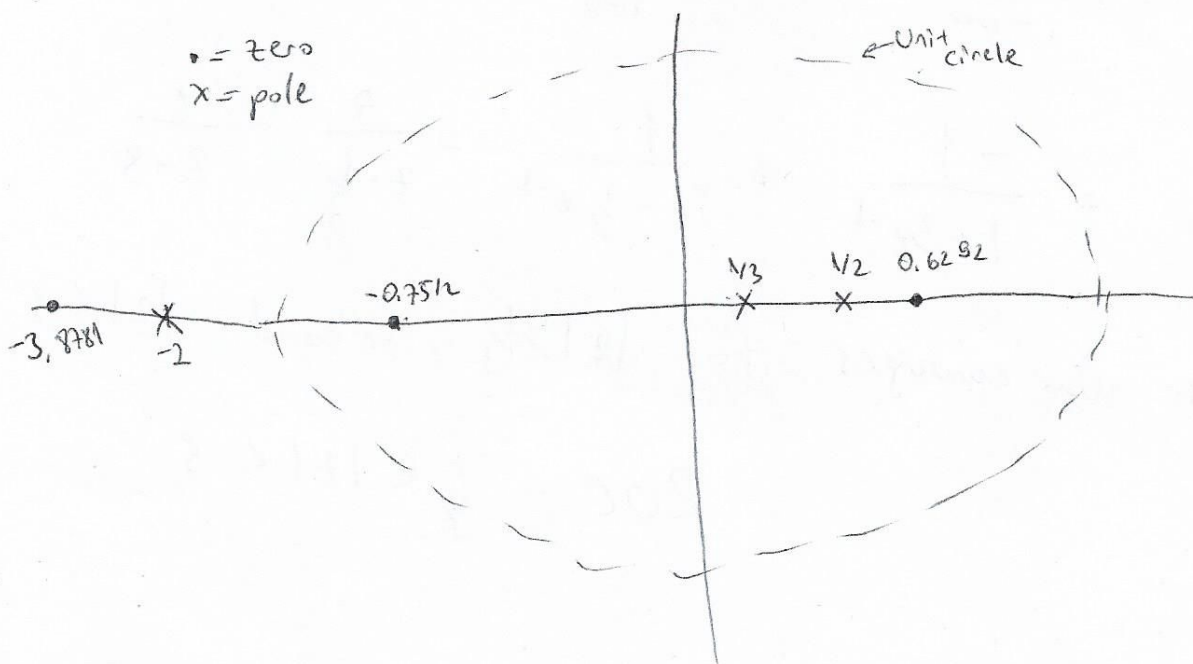
$$= 2 - \frac{z}{z-2} = \frac{z-4}{z-2}$$

$$\text{ROC: } |z| > 2 //$$

$$\begin{aligned} 8.2. \quad X(z) &= \frac{z^3 + 4z^2 - \frac{11}{6}}{z^3 + \frac{7}{6}z^2 - \frac{3}{2}z + \frac{1}{3}} \\ &= \frac{z^3 + 4z^2 - \frac{11}{6}}{(z - \frac{1}{3})(z - \frac{1}{2})(z+2)} \end{aligned}$$

$$= \frac{(z + 3.8181)(z + 0.7512)(z - 0.6292)}{(z - \frac{1}{3})(z - \frac{1}{2})(z+2)}$$

(a)



(3)

(b) (i) If ROC is $|z| < \frac{1}{3}$, all the poles are anti-causal.

(ii) If ROC is $|z| > \frac{1}{3}$, all the poles are causal.

$$(c) \quad X(z) = \frac{z^3 + 4z^2 - \frac{11}{6}}{(z - \frac{1}{3})(z - \frac{1}{2})(z + 2)}$$

By applying partial fraction method;

$$\frac{X(z)}{z} = \frac{z^3 + 4z^2 - \frac{11}{6}}{z(z - \frac{1}{3})(z - \frac{1}{2})(z + 2)} = \frac{A_1}{z - \frac{1}{3}} + \frac{A_2}{z - \frac{1}{2}} + \frac{A_3}{z + 2} + \frac{A_0}{z}$$

$$\text{For } A_0; \left. \frac{z^3 + 4z^2 - \frac{11}{6}}{(z - \frac{1}{3})(z - \frac{1}{2})(z + 2)} \right|_{z=0} = -\frac{11}{2}$$

$$\text{For } A_1; \left. \frac{z^3 + 4z^2 - \frac{11}{6}}{z(z - \frac{1}{2})(z + 2)} \right|_{z=\frac{1}{3}} = \frac{73}{7}$$

$$\text{For } A_2; \left. \frac{z^3 + 4z^2 - \frac{11}{6}}{z(z - \frac{1}{3})(z + 2)} \right|_{z=\frac{1}{2}} = \frac{-17}{5}$$

$$\text{For } A_3; \left. \frac{z^3 + 4z^2 - \frac{11}{6}}{z(z - \frac{1}{2})(z - \frac{1}{3})} \right|_{z=-2} = -\frac{37}{70}$$

(4)

$$X(z) = \frac{73}{7} \frac{z}{z - \frac{1}{3}} - \frac{17}{5} \frac{z}{z - \frac{1}{2}} - \frac{37}{70} \frac{z}{z+2} - \frac{11}{2}$$

(i) $|z| < \frac{1}{3}$, for anti causal case

$$x(n) = \frac{-73}{7} \left(\frac{1}{3}\right)^n u(-n-1) + \frac{17}{5} \left(\frac{1}{2}\right)^n u(-n-1) + \frac{37}{70} (-2)^n u(-n-1) + \frac{11}{2} \delta(-n-1)$$

(ii) $|z| > 2$, for causal case

$$x(n) = \frac{73}{7} \left(\frac{1}{3}\right)^n u(n) - \frac{17}{5} \left(\frac{1}{2}\right)^n u(n) - \frac{37}{70} (-2)^n u(n) - \frac{11}{2} \delta(n)$$

8.3. (c) $x(n] = \underbrace{n \left(\frac{1}{2}\right)^n}_{x_1(n)} + \underbrace{(n-1) \left(\frac{1}{3}\right)^n}_{x_2(n)}$

$$\begin{aligned} \left(\frac{1}{2}\right)^n &\xleftrightarrow{\text{z. transform}} \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{z}{z - \frac{1}{2}} \\ n \left(\frac{1}{2}\right)^n &\xleftrightarrow{\text{z. transform}} \frac{\frac{1}{2}z}{\left(z - \frac{1}{2}\right)^2} \end{aligned} \quad \left. \vphantom{\begin{aligned} \left(\frac{1}{2}\right)^n \\ n \left(\frac{1}{2}\right)^n \end{aligned}} \right\} \text{Differentiation}$$

$$X_1(z) = \frac{\frac{1}{2}z}{\left(z - \frac{1}{2}\right)^2}$$

(5)

$$\left(\frac{1}{3}\right)^n \xleftrightarrow{\text{z transform}} \frac{1}{1 - \frac{1}{3}z^{-1}} = \frac{z}{z - \frac{1}{3}}$$

DIFF.

$$n \left(\frac{1}{3}\right)^n \xleftrightarrow{\text{z transform}} \frac{\frac{1}{3}z}{\left(z - \frac{1}{3}\right)^2}$$

$$(n-1) \left(\frac{1}{3}\right)^{n-1} \xleftrightarrow{\text{z transform}} z^{-1} \frac{\frac{1}{3}z}{\left(z - \frac{1}{3}\right)^2} = \frac{1/3}{\left(z - \frac{1}{3}\right)^2}$$

$$x_2(n) = (n-1) \left(\frac{1}{3}\right)^{n-1} \cdot \frac{1}{3}$$

$$X_2(z) = \frac{\frac{1}{9}}{\left(z - \frac{1}{3}\right)^2}$$

$$X(z) = X_1(z) + X_2(z)$$

$$X(z) = \frac{\frac{1}{2}z}{\left(z - \frac{1}{2}\right)^2} + \frac{\frac{1}{9}}{\left(z - \frac{1}{3}\right)^2} //$$

8.12. $y(n) = 6 \left(-\frac{1}{4}\right)^n u(n) - 6 \left(-\frac{1}{3}\right)^n u(n)$

$$x(n) = u(n) + \left(-\frac{1}{2}\right)^n u(n)$$

(6)

$$Y(z) = \frac{6z}{z + \frac{1}{4}} - \frac{6z}{z + \frac{1}{3}} = \frac{\frac{1}{2}z}{(z + \frac{1}{4})(z + \frac{1}{3})}$$

$$X(z) = \frac{z}{z-1} + \frac{z}{z + \frac{1}{2}} = \frac{2z(z - \frac{1}{2})}{(z-1)(z + \frac{1}{2})}$$

$$H(z) = \frac{Y(z)}{X(z)} = \left(\frac{\frac{2z(z - \frac{1}{4})}{(z-1)(z + \frac{1}{2})}}{\frac{\frac{1}{2}z}{(z + \frac{1}{4})(z + \frac{1}{3})}} \right)^{-1} = \left(\frac{2z(z - \frac{1}{4})}{(z-1)(z + \frac{1}{2})} \cdot \frac{(z + \frac{1}{4})(z + \frac{1}{3})}{\frac{1}{2}z} \right)^{-1}$$

$$H(z) = \frac{(z-1)(z + \frac{1}{4})}{4(z + \frac{1}{4})(z + \frac{1}{3})(z - \frac{1}{4})}$$

$$H(z) = \frac{z^2 - \frac{1}{2}z - \frac{1}{2}}{4z^3 + \frac{4}{3}z^2 - \frac{1}{4}z - \frac{1}{12}}$$

(9)

$$H(z) = \frac{z^{-1} - \frac{1}{2}z^{-2} - \frac{1}{2}z^{-3}}{4 + \frac{4}{3}z^{-1} - \frac{1}{4}z^{-2} - \frac{1}{12}z^{-3}}$$

(7)

$$(b) H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} - \frac{1}{2}z^{-2} - \frac{1}{2}z^{-3}}{4 + \frac{4}{3}z^{-1} - \frac{1}{4}z^{-2} - \frac{1}{12}z^{-3}}$$

$$Y(z) \left(4 + \frac{4}{3}z^{-1} - \frac{1}{4}z^{-2} - \frac{1}{12}z^{-3} \right) = X(z) \left(z^{-1} - \frac{1}{2}z^{-2} - \frac{1}{2}z^{-3} \right)$$

↓

$$4y(n) + \frac{4}{3}y(n-1) - \frac{1}{4}y(n-2) - \frac{1}{12}y(n-3) = x(n-1) - \frac{1}{2}x(n-2) - \frac{1}{2}x(n-3)$$

Oppenheim Book:

10.2. $\left(\frac{1}{5}\right)^n u(n) \xleftrightarrow{z.\text{ transform}} \frac{1}{1 - \frac{1}{5}z^{-1}}$

$$x[n] = \left(\frac{1}{5}\right)^3 \left(\frac{1}{5}\right)^{n-3} u(n-3)$$

$$X(z) = \frac{1}{125} \cdot z^{-3} \frac{1}{1 - \frac{1}{5}z^{-1}} = \frac{z^{-2}}{125(z - \frac{1}{5})} //$$

$$\text{ROC: } |z| > \frac{1}{5}$$

(8)

10.4.
$$X(z) = \sum_{n=-\infty}^0 \left(\frac{1}{3}\right)^n \cos\left(\frac{\pi}{4}n\right) z^{-n}$$

$$= \frac{1}{2} \sum_{n=-\infty}^0 \left(\frac{1}{3}\right)^n e^{j\frac{\pi}{4}n} z^{-n} + \frac{1}{2} \sum_{n=-\infty}^0 \left(\frac{1}{3}\right)^n e^{-j\frac{\pi}{4}n} z^{-n}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{-n} e^{-j\frac{\pi}{4}n} z^n + \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{-n} e^{j\frac{\pi}{4}n} z^n$$

$$= \frac{1}{2} \frac{1}{1 - 3e^{-j\pi/4}z} + \frac{1}{2} \frac{1}{1 - 3e^{j\pi/4}z} \quad |z| < \frac{1}{3}$$

The poles are at $z = \frac{1}{3}e^{j\pi/4}$ and $z = \frac{1}{3}e^{-j\pi/4}$ //

10.9. Using partial-fraction expansion

$$X(z) = \frac{2/9}{1-z^{-1}} + \frac{7/9}{1+2z^{-1}}, \quad |z| > 2.$$



$$x(n) = \frac{2}{9} u[n-1] + \frac{7}{9} (-2)^n u[n]$$

9

$$10.21. (g) \quad x[n] = \underbrace{2^n u[-n]}_{x_1[n]} + \underbrace{\left(\frac{1}{4}\right)^n u[n-1]}_{x_2[n]}$$

$$X_1(z) = \sum_{n=-\infty}^0 2^n z^{-n}$$

$$= \sum_{n=0}^{\infty} 2^{-n} z^n = \frac{1}{1 - \frac{1}{2}z}, \quad |z| < 2$$

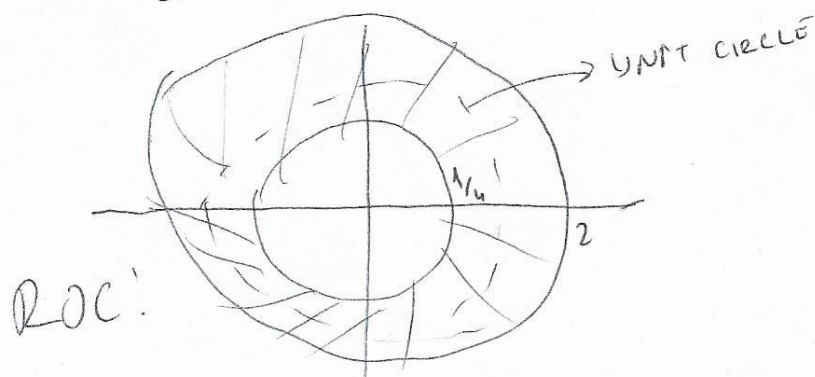
$$\left(\frac{1}{4}\right)^n u[n] \xleftrightarrow{z\text{-transform}} \frac{1}{1 - \frac{1}{4}z^{-1}}, \quad |z| > \frac{1}{4}$$

$$X_2[n] = \left(\frac{1}{4}\right)^{n-1} u[n-1] \cdot \frac{1}{4}$$

$$X_2(z) = \frac{1}{4} \cdot z^{-1} \cdot \frac{1}{1 - \frac{1}{4}z^{-1}}, \quad |z| > \frac{1}{4}$$

$$X(z) = X_1(z) + X_2(z)$$

$$= \frac{z^{-1}}{z^{-1} - 1/2} + \frac{z^{-1}/4}{1 - (1/4)z^{-1}}, \quad \frac{1}{4} < |z| < 2$$



(10)

The Fourier transform exists because the ROC includes the unit circle.

$$(h) \quad x[n] = \left(\frac{1}{3}\right)^{n-2} u[n-2]$$

$$\left(\frac{1}{3}\right)^n u[n] \xleftrightarrow{z\text{-transform}} \frac{1}{1 - \frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$

$$\left(\frac{1}{3}\right)^{n-2} u[n-2] \xleftrightarrow{z\text{-transform}} z^{-2} \frac{1}{1 - \frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$

$$X(z) = \frac{z^{-2}}{1 - \frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$

The Fourier transform exists because the ROC includes the unit circle.

Soliman

8.3)

$$a) \quad z[a^n \sin \Omega_0 n] = \frac{az \sin \Omega_0}{z^2 - 2az \cos \Omega_0 + a^2}$$

$$\begin{aligned} z[na^n \sin \Omega_0 n] &= -z \frac{\partial}{\partial z} \left[\frac{az \sin \Omega_0}{z^2 - 2(a \cos \Omega_0)z + a^2} \right] \\ &= \frac{a \sin \Omega_0 z^3 - a^3 \sin \Omega_0 z}{[z^2 - 2az \cos \Omega_0 + a^2]^2} \end{aligned}$$

$$d) \quad X(z) = z^{-2} + \frac{z}{(z-1)^2}$$

Oppenheim

10.21)

$$a) \quad x(n) = \delta(n+5)$$

$$X(z) = z^5 \quad \forall z.$$

The Fourier transform exist because the Roc includes the unit Circle.

$$b) \quad x(n) = \delta(n-5).$$

$$X(z) = z^{-5} \quad \forall z \text{ except at } 0.$$

Fourier transform exists.

$$c) \quad x(n) = (-1)^n u(n). \quad X(z) = \sum_{n=0}^{\infty} (-1)^n z^{-n} = \frac{1}{1+z^{-1}}; |z| > 1$$

Fourier transform does not exist.

$$d) \quad x(n) = \left(\frac{1}{2}\right)^{n+1} u(n+3).$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-3}^{\infty} \left(\frac{1}{2}\right)^{n+1} z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+2} z^{-n+3}$$

$$= \frac{4z^3}{\left(1 - \frac{1}{2}z^{-1}\right)} \quad |z| > \frac{1}{2}.$$

Fourier transform exists because the ROC includes the unit circle.

$$e) \quad x(n) = \left(-\frac{1}{3}\right)^n u[-n-2].$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-2}^{-1} \left(-\frac{1}{3}\right)^n z^{-n}$$

$$= \sum_{n=2}^{\infty} \left(-\frac{1}{3}\right)^{-n} z^n = \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^{-n-2} z^{n+2}$$

$$= \frac{9z^2}{(1+3z)} \quad |z| < \frac{1}{3}.$$

Fourier transform does not exist because the ROC does not include the unit circle.

f)

$$x(n) = \left(\frac{1}{4}\right)^n u(-n+3).$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^3 \left(\frac{1}{4}\right)^n z^{-n}$$

$$= \sum_{n=-3}^{\infty} \left(\frac{1}{4}\right)^{-n} z^{-n}.$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{-n+3} z^{-n+3}.$$

$$= \frac{1}{64} \frac{z^{-3}}{(1-4z)} , \quad |z| < \frac{1}{4}.$$

Fourier transform does not exist.

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